Approach of background metric expansion to a new metric ansatz for gauged and ungauged Kaluza-Klein supergravity black holes

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In a previous paper [S.Q. Wu, Phys. Rev. D **83**, 121502(R) (2011)], a new kind of metric ansatz was found to fairly describe all already-known black hole solutions in the ungauged Kaluza-Klein (KK) supergravity theories. That metric ansatz somewhat resembles to the famous Kerr-Schild (KS) form, but it is different from the KS one in two distinct aspects. That is, apart from a global conformal factor, the metric ansatz can be written as a vacuum background spacetime plus a "perturbation" modification term, the latter of which is associated with a timelike geodesic vector field rather than a null geodesic congruence in the usual KS ansatz. Replacing the flat vacuum background metric by the (anti-)de Sitter [(A)dS] spacetime, the general rotating charged KK-(A)dS black hole solutions in all higher dimensions have been successfully constructed and put into a unified form. In this paper, we shall study this novel metric ansatz in detail, aiming at achieving some inspiration as to the construction of rotating charged AdS black holes with multiple charges in other gauged supergravity theories. We find that the traditional perturbation expansion method often successfully used in the KS form is no longer useful in our new ansatz, since here no good parameter can be chosen as a suitable perturbation indicator. In order to investigate the metric properties of the general KK-AdS solutions, in this paper we devise a new effective method, dubbed the background metric expansion method, which can be thought of as a generalization of the perturbation expansion method, to deal with the Lagrangian and all equations of motion. In addition to two previously known conditions, namely the timelike and geodesic properties of the vector, we get three additional constraints via contracting the Maxwell and Einstein equations once or twice with this timelike geodesic vector. In particular, we find that these are a simpler set of sufficient conditions to determine the vector and the dilaton scalar around the background metric, which is helpful in obtaining new exact solutions. With these five simpler equations in hand, we rederive the general rotating charged KK-(A)dS black hole solutions with spherical horizon topology and obtain new solutions with planar topology in all dimensions. It turns out that the overall calculations in finding the solution to the KK gauged supergravity can be reduced considerably, compared to the previous process, by directly solving all the field equations. It is then shown that the rotating charged KK-AdS black hole solutions can be further generalized by introducing one or two arbitrary constants, while the black hole solutions with the planar AdS background metric in all higher dimensions are newly obtained.

I. INTRODUCTION

It has been known for a long time that Einstein's gravitational field equations are such a very complicated system of nonlinearly coupled partial differential equations that finding a rotating exact solution to them is rather difficult. One of the frequently used approaches to this problem is to assume an appropriate metric form for the unknown line element which is inspired from a known solution in order to simplify the subsequent calculations. A well-known example of this is provided by the Kerr solution [1], which can be cast into the famous Kerr-Schild (KS) form [2–4]

$$g_{ab} = \eta_{ab} + 2H\,k_a k_b\,, \qquad (1.1)$$

where the vector k^a is null and geodesic congruence with respect to the flat background metric η_{ab}, and H is a scalar function. Due to the fact that many interesting properties are shared by this family of the metric ansatz [5] and their applications result in a substantial simplification of the field equations, a variety of generalizations

of the KS metric form have been accomplished during the past decades. Below we present a brief outline of the main developments that have been achieved on the generalizations of the Kerr-Schild metrical structure.

(a) The generalized Kerr-Schild ansatz, in which the background metric is replaced by arbitrary spacetimes $(\eta \to \bar{g})$, was proposed and analyzed in Refs. [6, 7]. Many previous studies have shown that the Einstein field equations become linear within the generalized Kerr-Schild ansatz for vacuum [8] and nonvacuum [9, 10] spacetimes.

(b) The double Kerr-Schild metric, namely

$$g_{ab} = \bar{g}_{ab} + 2P\,k_a k_b + 2Q\,l_a l_b + 4R\,k_{(a}l_{b)} \qquad (1.2)$$

with $\bar{g}_{ab}k^a k^b = \bar{g}_{ab}l^a l^b = \bar{g}_{ab}k^a l^b = 0$, where P, Q, and R are three scalar functions, was introduced [11–15] in the context of complex relativity in dimension $D = 4$. It is often misunderstood that in Lorentzian signature, two such null vectors must be proportional. However, this is not always the case if one considers the *complex* Riemannian space with Lorentzian signature. To explain this point, an explicit example for the four-dimensional Kerr-

Newman-Unti-Tamburino-AdS (Kerr-NUT-AdS) metric is provided in the Appendix. Further observations indicate that such a general class of metric ansatz could be further generalized to the double Kerr-Schild metric [16] in $D > 4$ and the multi-Kerr-Schild metric [17] in $D \geqslant 4$, where the orthogonality properties still hold.

(c) An earlier generalization of the Kerr-Schild ansatz by introducing a nonzero cosmological constant in four-dimensional spacetimes was considered in [18]. Further studies [19] showed that all static spherically symmetric vacuum spacetimes with or without the cosmological constant can be described by conformal Kerr-Schild metrics [20].

(d) Higher-dimensional generalization of the Kerr-Schild metric was firstly utilized by Myers and Perry [21] to construct exact, asymptotically flat vacuum solutions of rotating black holes in all higher dimensions $D > 4$. Recently, rotating vacuum black holes with a nonzero cosmological constant in higher dimensions were successfully constructed in [22, 23] by simply replacing the flat background metric of the higher-dimensional KS form by the pure (A)dS spacetime. Moreover, further investigations have demonstrated that the background metric can be replaced by other asymptotically, locally flat spacetimes such as those with the NUT charges; an important example of this includes the NUT extension [24] of rotating black holes in (A)dS spacetimes. General properties of higher-dimensional Ricci-flat and (A)dS Kerr-Schild metric mentioned above were studied recently in [25–28].

(e) One recent extension of the original KS form is named the extended Kerr-Schild ansatz in [29, 30], where the metric of rotating charged black hole, namely the CCLP [31] spacetime found in $D = 5$ minimal gauged supergravity, can be redescribed in the framework of Kerr-Schild formalism as

$$g_{ab} = \bar{g}_{ab} + H \, k_a k_b + V \, (k_a l_b + l_a k_b) \,, \qquad (1.3)$$

in which H and V are two scalar functions to be specified; k^a is again a null vector, and the vector l^a is spacelike and orthogonal to k^a with respect to $\bar{g}_{ab}$, a flat or (A)dS background metric.

The general properties such as geodesic and optical properties of the null congruence and Weyl types of this kind of metric have been investigated in [32, 33]. In addition, it is worth pointing out that a limited case for the related extension of the Kerr-Schild ansatz was studied in [34], where the background is Ricci flat in $D = 4$.

In our previous paper [35], a new kind of metric ansatz was found to satisfactorily describe all already-known black hole solutions in the ungauged Kaluza-Klein (KK) supergravity theories, which can be written in a new unified form in all higher $D > 4$ dimensions by

$$g_{ab} = H^{\frac{1}{D-2}} \left(\bar{g}_{ab} + \frac{2m}{UH} k_a k_b \right),$$
$$g^{ab} = H^{\frac{-1}{D-2}} \left(\bar{g}^{ab} - \frac{2m}{U} k^a k^b \right), \qquad (1.4)$$

where the background metric is a flat one. This metric ansatz is different from the entire above-mentioned generalization of the original KS ansatz. It somewhat resembles to the famous Kerr-Schild form, but there are significant differences from the KS one in two distinct aspects, that is, apart from a common conformal factor, the vector k^a is no longer null but now it is timelike with respect to the background metric. The timelike vector field k^a is geodesic and its norm with respect to the background metric depends on the charge parameter: $\bar{g}_{ab} k^a k^b = -s^2$. It should be noticed that in the uncharged case, the conformal factor becomes unity and the vector k^a becomes null; then our new metric ansatz exactly reduces to the original KS metric (1.1).

It has further been observed in Ref. [35] that one can adopt the pure (A)dS spacetimes as the background metric and find the general rotating charged Kaluza-Klein (A)dS black hole solutions with a single electric charge and arbitrary angular momenta as the exact solutions to the Einstein-Maxwell-dilaton theory described by the following Lagrangian ($\mathcal{F} = dA$)

$$\mathcal{L} = \sqrt{-g} \Big\{ R - \frac{1}{4}(D-1)(D-2)(\partial \Phi)^2 - \frac{1}{4} e^{-(D-1)\Phi} \mathcal{F}^2$$
$$+ g^2 (D-1) \big[(D-3) e^{\Phi} + e^{-(D-3)\Phi} \big] \Big\}. \qquad (1.5)$$

For the sake of later simplicity, the spacetime of this general form shall be briefly called the stringy Kerr-Schild or sKS metric, since its metric structure has some relation to the well-known Kerr-Schild form, and it is universal for almost all of the charged black hole solutions already known in gauged supergravity theories. As such, the underlying metric structure of our sKS form can be thought of as the most meaningful generalization of the Kerr-Schild ansatz until now. In addition, it should also be mentioned that the $D = 4$ sKS metric can be expressed as a form similar to those proposed by Yilmaz [36] and later by Bekenstein [37]. In addition, many studies have brought out that the Gordon metric can be further applied to massive and bimetric theory (see [38–40]), the metric structure of which also resembles the generalized sKS metric form specifically in $D = 4$.

The main subject of this paper is to investigate the general properties of the field equations for the sKS metric, since such an important theoretical analysis had not been delivered before in any previous work. Moreover, our motivation of this study is to see whether the results of such a theoretical analysis could be helpful in obtaining new exact solutions with the help of the sKS form and in achieving some insights on constructing exact solutions to other gauged supergravity theories.

The organization of this paper is outlined as follows. To begin with, in Sec. II we will show that it is infeasible to analyze the sKS ansatz by the usual method of perturbation expansion. Although this method proved to be inappropriate for our aim, one can still get a little inspiration from it. As a replacement, we therefore put forward a new method, named the background metric expansion,

which can be viewed as the generalization of the previous one. In the actual derivation, it is much more cumbersome for the sKS metric to express the Ricci tensor and field equations, so it is convenient and in fact necessary to use the computer algebra system Cadabra [42, 43], which allows us to perform analytically symbolic calculations in arbitrary unspecified dimensions. In Sec. III, our new method is used to obtain the geodesic property of the vector k^a with respect to the background metric, and a simpler set of sufficient conditions of the field equations around the background metric are deduced. With these results in hand, in Sec. IV we assume a vector $\tilde{K}$ to be timelike and geodesic and then solve the simpler set of field equations around the pure AdS background spacetime. Consequently, we obtain an extended version of rotating charged KK-AdS solutions with one or two arbitrary constants (one for $\epsilon = 1$, two for $\epsilon = 0$). As another verification of the effect of our method, we further obtain new exact solutions with planar topology by replacing the background metric as a planar AdS metric [41]. In Sec. V, we summarize our results and discuss the prospect of the analysis, for which it would be very helpful for finding new exact black hole solutions.

II. INFEASIBILITY OF TRADITIONAL PERTURBATION METHOD FOR THE SKS ANSATZ

In this section, we present the usual formalism of perturbation expansion that goes into these calculations for analogy and explain why the ordinary procedure is limited and inappropriate for our new sKS form.

For simplicity, we use the dilaton scalar Φ to reexpress the metric tensors and the gauge potential as [35]

$$
\begin{aligned}
g_{ab} &= e^{-\Phi}\bar{g}_{ab} + \lambda\big[e^{-\Phi} - e^{(D-3)\Phi}\big]k_a k_b\,, \\
g^{ab} &= e^{\Phi}\bar{g}^{ab} + \lambda\big[e^{\Phi} - e^{-(D-3)\Phi}\big]k^a k^b\,, \\
A_a &= \sqrt{\lambda}\big[1 - e^{(D-2)\Phi}\big]k_a\,, \quad \Phi = \frac{-1}{D-2}\ln(H)\,,
\end{aligned}
\tag{2.6}
$$

where the vector k^a is a timelike geodesic congruence with respect to the AdS background metric $\bar{g}_{ab}$ and satisfies $k_a = \bar{g}_{ab}k^b$, $k_a k^a = \bar{g}_{ab}k^a k^b = -1$. The λ is inserted here as a dimensionless parameter that would take the value $\lambda = 1$ finally. When $\lambda = 1$, we have for the full metric tensor the following properties:

$$
\begin{aligned}
g_{ab}k^b &= e^{(D-3)\Phi}k_a\,, \quad g_{ab}k^a k^b = -e^{(D-3)\Phi}\,, \\
g^{ab}k_b &= e^{(3-D)\Phi}k^a\,, \quad g^{ab}k_a k_b = -e^{(3-D)\Phi}\,.
\end{aligned}
\tag{2.7}
$$

We note that the timelike vector k^a also satisfies the geodesic property: $k^a\bar{\nabla}_a k^b = 0$ with the specific spacetime, where $\bar{\nabla}_a$ denotes the covariant derivative operator compatible with the background metric $\bar{g}_{ab}$. Although this geodesic property is very helpful to simplify our computations during the subsequent perturbation process, we will only apply the essential relations (2.6) and the

timelike condition to do perturbational analysis in an attempt to figure out the most universal properties of the sKS ansatz.

The curvature of the sKS metric as well as other useful quantities can be computed in terms of the curvature of the background metric $\bar{g}_{ab}$ and the background covariant derivative of the vectors k^a. The action of the full covariant derivative on a vector can be written as $\nabla_a v^b = \bar{\nabla}_a v^b + C^b{}_{ac}v^c$, in which the connection $C^c{}_{ab}$ is given by

$$
C^c{}_{ab} = \frac{1}{2}g^{cd}\big(\bar{\nabla}_a g_{bd} + \bar{\nabla}_b g_{ad} - \bar{\nabla}_c g_{ab}\big)\,;
\tag{2.8}
$$

then the Ricci tensor of g_{ab} is related to that of $\bar{g}_{ab}$ by

$$
R_{ab} = \bar{R}_{ab} - 2\bar{\nabla}_{[a}C^c{}_{c]b} + 2C^e{}_{a[b}C^c{}_{c]e}\,.
\tag{2.9}
$$

(See also [5].) The determinant of the full metric of sKS form is related to the background one by $\sqrt{-g} = e^\Phi\sqrt{-\bar{g}}$; hence we have an identity $C^b{}_{ab} = -\bar{\nabla}_a\Phi$.

After using Cadabra [42, 43] software to undertake the tedious calculations, we can write the connection coefficients and the Ricci tensor containing terms quadratic in connection coefficients as a sum over contributions at different powers in λ as follows:

$$
\begin{aligned}
C^c{}_{ab} &= \sum_{k=0}^{2}\lambda^k C^{c(k)}{}_{ab} \\
&= \frac{1}{2}\bar{g}_{ab}\bar{\nabla}^c\Phi - \bar{g}^c{}_a\bar{\nabla}_b\Phi + \sum_{k=1}^{2}\lambda^k C^{c(k)}{}_{ab}\,, \\
R_{ab} &= \sum_{l=0}^{4}\lambda^l R^{(l)}_{ab}\,.
\end{aligned}
\tag{2.10}
$$

Based upon these expressions, the Einstein equation $E_{ab} = 0$ for the KK-AdS spacetime (1.5) in terms of the background metric $\bar{g}_{ab}$ can be represented as $E_{ab} = \sum_{n=0}^{4}\lambda^n E^{(n)}_{ab}$. Note that, in the uncharged case ($\Phi = 0 = A_a$), the metric structure (2.6) and all the above equations reduce to the original Kerr-Schild form.

In Eq. (2.10), we have only considered the expansion of the Ricci tensor R_{ab} in terms of the parameter λ. One can also work with the mixed tensor $R^a{}_b$. Unlike the case of the standard Kerr-Schild form, the tensor $R^a{}_b$ now still contains nonlinear terms in λ, just like R_{ab}. This is because the vector k^a is a timelike, rather than a null vector. Since both R_{ab} and $R^a{}_b$ contain nonlinear terms in λ, it is of no priority to consider the component with the mixed indices. For this reason, we prefer considering the full covariant component in this paper. If one would like to work with the mixed components, then it is easy to find that they are just a recombination of the covariant components.

To proceed further, it is facilitated by directly considering the contracted equation $E_{ab}k^a k^b = 0$. Now one would naively expect the corresponding expressions for $E^{(n)}_{ab}k^a k^b$ in front of λ^n at each order to vanish just as

in the previous case considered in [30]. Our computation shows that the contribution from the fourth order $E^{(4)}_{ab}k^a k^b$ vanishes identically, while at order λ^3 it reads

$$
\begin{aligned}
E^{(3)}_{ab}k^a k^b = & -\frac{(1-\gamma)^4}{2(D-2)\gamma^2}(\bar{D}k_a)\bar{D}k^a \\
& +\frac{3\gamma-5+D}{2}\left(1-\gamma^{-1}\right)^2(\bar{\nabla}^a\phi)\bar{D}k_a \quad (2.11) \\
& +\frac{\alpha}{4\gamma^2}(1-\gamma)\big[\bar{D}\Phi+(\bar{\nabla}\Phi)^2\big],
\end{aligned}
$$

where we denote $\gamma = e^{-(D-2)\Phi}$ and $\alpha = \gamma^2+2(D-5)\gamma+D^2-8D+13$, while $\bar{D} = k^a\bar{\nabla}_a$ is the background covariant derivative taken along the null vector k^a. Obviously, once we consider the geodesic property $\bar{D}k_a = 0$, the expression (2.11) vanishes identically if and only if the condition $\bar{D}\Phi+(\bar{\nabla}\Phi)^2 = 0$ is satisfied, where we have taken a nonzero scalar function Φ into account (and thus $\gamma \neq 1$, $\alpha \neq 0$, and $3\gamma \neq D-5$). However, it is clear to check that this condition is inconsistent with the explicit KK-AdS solutions in [35], indicating that this is a meaningless condition.

We have also attempted to place the perturbation factor λ in different positions of the metric ansatz, but still failed to move forward. Actually, it not only shows that the insertion of the perturbation factor λ is not suitable, but also reveals that this insertion is irrational and unreasonable for the sKS ansatz. In fact, the underlying reason is that there exists no perturbation parameter (neither the mass nor the charge) as an appropriable indicator to do the corresponding perturbation analysis here, unlike the cases that are successful for the original KS form and the extended Ks ansatz, where the mass parameter can be treated as a perturbation parameter. Due to the feature of the conformal factor structure of the sKS metric ansatz, obviously direct application of the ordinary perturbation expansion fails for the present situation; therefore, as an alternative program, a new analysis method is needed.

III. NEW METHOD OF BACKGROUND METRIC EXPANSION FOR THE SKS METRIC ANSATZ

In this section, we will now propose a new background metric expansion method towards analyzing the sKS ansatz for the Einstein-Maxwell-dilaton system, which can be seen as a generalization of the ordinary perturbation expansion method. In this new method, we will synthetically consider the entire expansions of the Lagrangian and all the field equations around the background spacetimes, in terms of the background metric and the background covariant derivatives, not just from the viewpoint of perturbation expansion by which each term can be sorted in terms of the different orders of λ. We then contract the Maxwell and Einstein equations once or twice with the timelike vector k^a to extract more

useful information. Here, we are interested in seeing what simplifications will occur and, in particular, what the implications of the resulting Lagrangian and field equations will make. From the alternative perspective, we would like to see whether the vector k^a and the scalar Φ satisfy some conditions that could be helpful for us to obtain new exact solutions. In doing so, we find that in addition to two previously known timelike and geodesic properties obeyed by the vector, one can get three additional constraint equations.

A. The Lagrangian expanded around the background metric

As shown in the previous section, there is no use in treating the λ as a perturbation parameter in the sKS metric ansatz; therefore, in the following we shall take $\lambda = 1$. Fortunately, one can still expand all expected quantities in terms of those of the background metric. Our staring point is the sKs ansatz (2.6) with $\lambda = 1$. With the help of Cadabra, the explicit expression of the Ricci scalar is given in terms of the background metric and the background covariant derivatives as

$$
\begin{aligned}
R = e^{\Phi}\Big\{ & \bar{R}+\bar{\Box}\Phi+(1-\gamma)\big[\bar{R}_{ab}k^a k^b-\bar{\nabla}^a\bar{\nabla}^b(k_a k_b)\big] \\
& -\frac{1}{4}(D-2)(D-3)(\bar{\nabla}\Phi)^2 \\
& +[1+(1-D)\gamma]k_a k_b\bar{\nabla}^a\bar{\nabla}^b\Phi \\
& +\frac{1}{2\gamma}(1-\gamma)^2\big[2(\bar{\nabla}^a k^b)\bar{\nabla}_{[a}k_{b]}+(\bar{D}k_a)\bar{D}k_b\big] \\
& +[3-D+(1-D)\gamma](\bar{D}k_a)\bar{\nabla}^a\Phi \\
& +[1+(3-2D)\gamma](\bar{D}\Phi)\bar{\nabla}_a k^a \\
& +\frac{1}{4}(D-2)[3-D+3(D-1)\gamma](\bar{D}\Phi)^2\Big\}.
\end{aligned}
$$

Continuing to calculate the Lagrangian expanded around the background metric, we could obtain the Lagrangian of the Einstein-Maxwell-dilaton system with respect to the background metric

$$
\begin{aligned}
\mathcal{L} = \sqrt{-\bar{g}}\Big\{ & \bar{R}+(1-\gamma)\big[\bar{R}_{ab}k^a k^b-(D-1)g^2\big] \\
& +g^2(D-1)(D-2)+\frac{1}{2}(1-\gamma)^2(\bar{D}k_a)\bar{D}k^a \\
& +\bar{\nabla}^a\big([1+(1-D)\gamma]k_a\bar{D}\Phi\big) \\
& -\bar{\nabla}^a\big[(1-\gamma)\bar{\nabla}^b(k_a k_b)\big]\Big\},
\end{aligned}
$$
$$(3.12)$$

in which the relation $\sqrt{-g} = e^{\Phi}\sqrt{-\bar{g}}$ has been used.

Now we assume the background metric $\bar{g}_{ab}$ is the pure AdS metric and substitute $\bar{R}_{ab} = -g^2(D-1)\bar{g}_{ab}$ into the

above expression; then we get

$$
\begin{aligned}
\mathcal{L} = \sqrt{-\bar{g}} \Big\{ & \bar{R} + g^2(D-1)(D-2) \\
& + \frac{1}{2}(1-\gamma)^2(\bar{D}k_a)\bar{D}k^a \\
& + \bar{\nabla}^a\big([1+(1-D)\gamma]k_a\bar{D}\Phi\big) \\
& - \bar{\nabla}^a\big[(1-\gamma)\bar{\nabla}^b(k_ak_b)\big] \Big\} \, .
\end{aligned}
\tag{3.13}
$$

This expression establishes the association between the Lagrangian for the full metric and that of the background metric. The variation of this Lagrangian with respect to the timelike vector k_a implies that $\bar{D}k^a$ must be a null vector: $(\bar{D}k_a)\bar{D}k^a = 0$. Since $\bar{D}k^a$ is also orthogonal to the timelike vector k^a, without loss of generality we can take

$$
\bar{D}k^a = 0 \, . \tag{3.14}
$$

This is equivalent to the statement that k^a is tangent to an affinely parameterized timelike geodesic congruence of the background metric. Besides, one can also arrive at the same conclusion by solving vector k^a within the field equations derived from the Lagrangian (3.13) with respect to $\bar{g}^{ab}$ and Φ. In the following, we will proceed to consider the expansions of all the equations of motion derived directly from the Lagrangian (1.5) by assuming that the background spacetime is the pure AdS metric and vector k^a is tangent to a congruence of affinely parameterized timelike geodesics with respect to the background metric to find out more properties or relations for further research.

B. Field equations expanded around the background metric

We now turn to consider all the field equations deduced directly from the Lagrangian (1.5) and expand them around the background metric. Calculating the variational derivatives of the Lagrangian (1.5) with respect to (g^{ab}, A_a, Φ), one can obtain the contracted Einstein equation

$$
\begin{aligned}
& R_{ab} - \frac{1}{4}(D-1)(D-2)(\nabla_a\Phi)\nabla_b\Phi \\
& \quad - \frac{\gamma}{2}e^{-\Phi}\Big[F_{ac}F_{bd}g^{cd} - \frac{g_{ab}}{2(D-2)}F^2\Big] \\
& \quad + g^2\frac{D-1}{D-2}(D-3+\gamma)e^{\Phi}g_{ab} = 0 \, ,
\end{aligned}
\tag{3.15}
$$

while the dilaton and gauge field equations are

$$
\Box\Phi + \frac{\gamma e^{-\Phi}}{2(D-2)}F^2 + 2g^2\frac{D-3}{D-2}(1-\gamma)e^{\Phi} = 0 \, ,
$$
$$
\nabla^a\big[\gamma e^{-\Phi}F_{ab}\big] = 0 \, .
\tag{3.16}
$$

To avoid the clutter expressions, we first introduce the following two notations:

$$
\begin{aligned}
\bar{V}_a \equiv & -2\bar{\nabla}^a\big\{(1-\gamma)\bar{\nabla}_{[a}k_{b]} + (D-2)\gamma(\bar{\nabla}_{[a}\Phi)k_{b]} \\
& + (1-\gamma)^2 k_{[a}\bar{D}k_{b]}\big\} \, ,
\end{aligned}
\tag{3.17}
$$

$$
\begin{aligned}
\bar{S} \equiv & \, e^{\Phi}\Big\{\bar{\Box}\Phi + \frac{2\gamma(1-\gamma)^2}{D-2}(\bar{\nabla}^a k^b)\bar{\nabla}_{[a}k_{b]} \\
& - (D-2)(\bar{\nabla}\Phi)^2 + \frac{\gamma^{-1}}{D-2}(1-\gamma)^3(\bar{D}k^a)\bar{D}k_a \\
& + (1-\gamma)\big[\bar{\nabla}^a(k_a\bar{D}\Phi) - (D-2)(\bar{D}\Phi)^2 \\
& - (\bar{D}k_a)\bar{\nabla}^a\Phi\big] + 2g^2\frac{D-3}{D-2}(1-\gamma)\Big\} \, .
\end{aligned}
\tag{3.18}
$$

Then the field equations (3.16) expanded around the background AdS metric convert to $\bar{V}_a = 0$ and $\bar{S} = 0$, respectively. After the expansion of the derivative operator, these two expressions are, however, still quite cumbersome. In order to proceed with some further simplifications, a simpler and more efficient approach that we now adopt is to consider the contraction of the field equations with respect to the vector k^a. Taking $\bar{V}_a k^a$ and $\bar{S}$, we find that they can be written as

$$
\begin{aligned}
\bar{V}_a k^a = & \, \gamma S_1 + \frac{\gamma}{D-2}S_2 + (1-\gamma)\bar{\nabla}_a(\bar{D}k^a) \\
& + 2k_a\bar{\nabla}_b\big[(1-\gamma)^2 k^{[a}\bar{D}k^{b]}\big] \, ,
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
(D-2)\bar{S} = & \, e^{\Phi}\big[(1-\gamma)S_1 + (D-2)S_2 \\
& - 2(D-2)(1-\gamma)\bar{\nabla}^a\Phi(\bar{D}k_a) \\
& + (\bar{D}k)^2\gamma^{-1}(1-\gamma)^3\big] \, ,
\end{aligned}
\tag{3.20}
$$

in terms of two simple notations

$$
\begin{aligned}
S_1 \equiv & \, 2\big[(\bar{\nabla}^a k^b)\bar{\nabla}_{[b}k_{a]} + g^2(D-3)\big](1-\gamma^{-1}) \\
& + (D-2)\gamma^{-1}\bar{\nabla}^b[\gamma k_b(\bar{D}\Phi)] \, ,
\end{aligned}
\tag{3.21}
$$

$$
\begin{aligned}
S_2 \equiv & \, \bar{\Box}\Phi - (D-2)(\bar{\nabla}\Phi)^2 \\
& - 2g^2\frac{D-3}{D-2}(1-\gamma^{-1}) \, .
\end{aligned}
\tag{3.22}
$$

Considering now the geodesic condition $\bar{D}k^a = 0$ and after substituting it into $\bar{V}_a k^a$ and $\bar{S}$, one can observe that they are, in fact, only the combinations of two scalars S_1 and S_2. This means that the sufficient conditions for the vanishing of the contracted equations $\bar{V}_a k^a$ and the scalar field equation $\bar{S}$ are that the expressions S_1 and S_2 must vanish simultaneously as well. One cannot extract any new useful information other than two simple scalar equations $S_1 = 0$ and $S_2 = 0$. In particular, note that the condition $S_2 = 0$ is independent of the vector k^a, but depends only on the dilaton scalar Φ.

Having extracted two simple conditions $S_1 = 0$ and $S_2 = 0$ from the contracted field equations (3.19) and the scalar field equation (3.20) around the background metric, it is necessary to see whether these two conditions are also sufficient for the field equation $V_a = 0$ with respect to the background metric. To this end, we now

rewrite the vector expression $\bar{V}_a$ in terms of S_1 and S_2 as

$$\bar{V}_a \equiv V_a - \frac{S_2}{D-2}\gamma k_a - 2\bar{\nabla}^b\big[(1-\gamma)^2 k_{[b}\bar{D}k_{a]}\big] \quad (3.23)$$

where

$$V_a \equiv \bar{\nabla}^b\big\{2(\gamma-1)\bar{\nabla}_{[b}k_{a]} + (D-2)\gamma k_b\bar{\nabla}_a\Phi\big\} \\ - (D-2)\gamma(\bar{\nabla}^b\Phi)\bar{\nabla}_b k_a + 2(D-3)(1-\gamma)g^2 k_a. \quad (3.24)$$

From the gauge field equation $\bar{V}_a = 0$, one can get a new equation $V_a = 0$. This condition, together with the geodesic equation $\bar{D}k^a = 0$ and $S_2 = 0$, is sufficient to ensure that the field equation $\bar{V}_a$ vanishes. It is also worth noting that S_1 is simply related to the vector V_a by $S_1 = \gamma k^a V_a$; thus $V_a = 0$ implies $S_1 = 0$ immediately. This means that $S_1 = 0$ is equivalent to $\bar{V}_a = 0$.

The remaining step is to consider the Einstein equation (3.15) and its contraction with the vector k_a once and twice. For this purpose, we first expand it around the background spacetime and convert it to the form denoted simply as $\bar{E}_{ab} = 0$, where

$$\bar{E}_{ab} \equiv \frac{1-\gamma^{-1}}{2(D-2)}\big[(D-2)(\gamma+D-3)k_ak_b - \bar{g}_{ab}\gamma\big]S_1 \\ - (D-2)\gamma\bar{\nabla}_{(a}\Phi\bar{D}k_{b)} + \frac{1}{2(D-2)\gamma^2}\big[(D-2)\gamma S_2 \\ + (1-\gamma)^3(\bar{D}k^c)\bar{D}k_c\big]\big[\bar{g}_{ab}\gamma + (\gamma+D-3)k_ak_b\big] \\ + (\gamma^{-1}-1)\big[(\gamma-3+D)k_ak_b(\bar{\nabla}^c\Phi)\bar{D}k_c \\ + (D-2)\gamma(\bar{D}\Phi)\bar{D}(k_ak_b)\big] - T_{ab} \\ + (\gamma^{-1}-1)V_{(a}k_{b)} + \frac{(1-\gamma)^2}{2\gamma}\big\{\bar{\nabla}^c[k_c\bar{D}(k_ak_b)] \\ + (\gamma-2)(\bar{D}k_a)\bar{D}k_b - (\bar{D}k_d)\bar{\nabla}^d(k_ak_b)\big\}, \quad (3.25)$$

has been recast into its contraction with k_a once and twice, while the non-contractible symmetric part is

$$T_{ab} \equiv 2(1-\gamma)\big\{g^2(\bar{g}_{ab}+k_ak_b) - \bar{\nabla}_{[c}k_{(a}\bar{\nabla}^c k_{b)}\big\} \\ + \bar{\nabla}^c\big[(1-\gamma)k_c\bar{\nabla}_{(a}k_{b)}\big]. \quad (3.26)$$

As is shown in the above, given the geodesic conditions $\bar{D}k_a = 0$, we have obtained three simpler equations: $V_a = 0$, $S_1 = 0$, and $S_2 = 0$. Using these conditions, a sufficient condition for $\bar{E}_{ab} = 0$ is that the tensor T_{ab} should vanish as well. Therefore, all the expanded field equations obtained by the background metric expansion method, $\bar{V}_a = 0$, $\bar{S} = 0$, and $\bar{E}_{ab} = 0$, will be satisfied if $\bar{D}k_a = 0$, $V_a = 0$, $S_2 = 0$, and $T_{ab} = 0$, which have been explicitly verified with the KK-AdS black hole solutions [35].

To summarize, we establish that for the stringy Kerr-Schild metrics with a geodesic timelike vector k^a, solving the field equations (3.15) and (3.16) could be reduced to solving straightforwardly the following three relative

simple equations around the background metric:

$$\bar{\Box}\Phi - (D-2)(\bar{\nabla}\Phi)^2 - 2g^2\frac{D-3}{D-2}(1-\gamma^{-1}) = 0, \quad (3.27)$$
$$\bar{\nabla}^b\big\{2(\gamma-1)\bar{\nabla}_{[b}k_{a]} + (D-2)\gamma k_b\bar{\nabla}_a\Phi\big\} \\ - (D-2)\gamma(\bar{\nabla}^b\Phi)\bar{\nabla}_b k_a \quad (3.28) \\ + 2(D-3)(1-\gamma)g^2 k_a = 0,$$
$$2(1-\gamma)\big\{g^2(\bar{g}_{ab}+k_ak_b) - \bar{\nabla}_{[c}k_{(a}\bar{\nabla}^c k_{b)}\big\} \\ + \bar{\nabla}^c\big[(1-\gamma)k_c\bar{\nabla}_{(a}k_{b)}\big] = 0. \quad (3.29)$$

Thus, the sufficient conditions for the sKS ansatz are the geodesic condition (3.14) on k^a, Eq. (3.27) on the dilaton scalar Φ, and the conditions (3.28) and (3.29) on k^a and Φ. In particular, the condition (3.27) depends only on the properties of the dilaton scalar Φ and this set of conditions is also satisfied spontaneously in the uncharged case ($\Phi = 0$). In a word, after assuming that the timelike vector k^a is geodesic, we find that all the field equations with respect to the background metric then can be reduced to the three conditions $S_2 = 0$, $V_a = 0$, and $T_{ab} = 0$ given above. Nevertheless, an open question is to see whether these wonderful results derived by the background metric expansion method for the sKS ansatz would find some as-yet-unknown exact solutions.

IV. APPLICATIONS: NEW KK-ADS SOLUTIONS WITH SPHERICAL AND PLANAR TOPOLOGY

Inspired by the observation that Eqs. (3.27), (3.28), and (3.29) around the background metric $\bar{g}_{ab}$ can be seen as the counterparts of the field equations (3.16), then one wonders naturally about whether there exists a new vector field that may be different from the known one in the given black hole solutions but still satisfies all the field equations. As a test of our results derived above, in the following we shall use the pure AdS background metrics with spherical and planar topology as two concrete examples to derive new exact solutions.

To present explicitly the general KK-AdS solutions in the below, we shall adopt conventions as those in [22, 35]. The dimension of spacetime is denoted as $D = 2N + 1 + \epsilon \geq 4$, with $N = [(D-1)/2]$ being the number of rotation parameters a_i and $2\epsilon = 1 + (-1)^D$. Let Φ_i be the N azimuthal angles in the N orthogonal spatial 2-planes, each with period 2π. The remaining spatial dimensions are parameterized by a radial coordinate r and by $N + \epsilon = n = [D/2]$ "direction cosines" μ_i obey the constraint $\sum_{i=1}^{N+\epsilon}\mu_i^2 = 1$, where $0 \leq \mu_i \leq 1$ for $1 \leq i \leq N$, and $-1 \leq \mu_{N+1} \leq 1$, $a_{N+1} = 0$ for even D. Moreover, shorthand notations $c = \cosh\delta$ and $s = \sinh\delta$ are used.

Now we would like to find where there exists a new vector field K tangent to an affinely parameterized timelike geodesic congruence of the AdS background metric,

which is assumed to have the general form

$$K = K_t(\tilde{\mu}_i)\, dt + K_r(r, \tilde{\mu}_i)\, dr + \sum_{i=1}^{N} K_{\phi_i}(\tilde{\mu}_i)\, d\phi_i\,, \quad (4.30)$$

where K_t, K_r, and K_{ϕ_i} are some unknown functions to be specified, and the notation $\tilde{\mu}_i \equiv \mu_1, \mu_2, ..., \mu_i (i = 1, ..., N + \epsilon)$ has been used.

A.   The spherical AdS background metric

Consider first the case of a spherical AdS background metric. Supposing that the vector K satisfies the equations (3.28) and (3.29) with respect to the pure AdS background metric given in [22, 35]

$$d\bar{s}^2 = -(1 + g^2 r^2) W\, dt^2 + F\, dr^2$$
$$+ \sum_{i=1}^{N+\epsilon} \frac{r^2 + a_i^2}{\chi_i} d\mu_i^2 + \sum_{i=1}^{N} \frac{r^2 + a_i^2}{\chi_i} \mu_i^2\, d\phi_i^2$$
$$- \frac{g^2}{(1 + g^2 r^2) W} \left(\sum_{i=1}^{N+\epsilon} \frac{r^2 + a_i^2}{\chi_i} \mu_i\, d\mu_i \right)^2, \quad (4.31)$$

where the scalar functions W and F are

$$W = \sum_{i=1}^{N+\epsilon} \frac{\mu_i^2}{\chi_i}\,, \quad F = \frac{r^2}{1 + g^2 r^2} \sum_{i=1}^{N+\epsilon} \frac{\mu_i^2}{r^2 + a_i^2}\,, \quad (4.32)$$

then the functions K_t, K_r, and K_{Φ_i} can be easily calculated. For a comparison with that presented in [35], we get the following new vector,

$$K = \sqrt{c^2 C_1 + g^2(\epsilon - 1) C_2}\, W\, dt + \sqrt{f(r)}\, F\, dr$$
$$- \sum_{i=1}^{N} \frac{\sqrt{a_i^2 \Xi_i + (\epsilon - 1) C_2}}{\chi_i} \mu_i^2\, d\phi_i \quad (4.33)$$

where $f(r) = c^2 C_1 - s^2(1 + g^2 r^2) - (\epsilon - 1) C_2 / r^2$, $\Xi_i = c^2 C_1 - s^2 \chi_i$, and $\chi_i = 1 - g^2 a_i^2$, while C_1 and C_2 are two arbitrary constants. If $C_1 = 1$ and $C_2 = 0$, then the solution reduces to that given in [35]. It should be pointed out that we have directly and explicitly checked that the above vector K, together with the full metric and the gauge potential 1-form

$$ds^2 = H^{1/(D-2)} \left(d\bar{s}^2 + \frac{2m}{UH} K^2 \right),$$
$$A = \frac{2ms}{UH} K\,, \quad \Phi = \frac{-1}{D-2} \ln(H)\,, \quad (4.34)$$

obeys the field equations derived from the Lagrangian (1.5) of the Einstein-Maxwell-dilaton theory. In the above, the scalar functions (U, H) are defined to be

$$U = r^\epsilon \sum_{i=1}^{N+\epsilon} \frac{\mu_i^2}{r^2 + a_i^2} \prod_{j=1}^{N} (r^2 + a_j^2)\,, \quad H = 1 + \frac{2ms^2}{U}\,. \quad (4.35)$$

B.   The planar AdS background metric

As an input, it has been assumed that the background metric $\bar{g}_{ab}$ is the pure AdS metric; we now take the planar AdS metric [41] as the background spacetime for further verification. Similarly, one can solve the timelike and geodesic vector $\tilde{K}$ assumed in (4.30) with respect to the planar AdS background metric satisfying Eqs. (3.28) and (3.29). Through a series of tedious calculations, we obtain the new planar AdS solutions as follows:

$$ds^2 = H^{1/(D-2)} \left(- g^2 r^2\, dt^2 + \frac{dr^2}{g^2 r^2} + r^2 d\Sigma_k^2 \right.$$
$$\left. + \frac{2ms^2}{r^{D-3} H} \tilde{K}^2 \right) \quad (4.36)$$
$$A = \frac{2ms^2}{r^{D-3} H} \tilde{K}\,, \quad \Phi = \frac{-1}{D-2} \ln(H)\,,$$

where $d\Sigma_k^2$ denotes the flat $k = (D-2)$-space unit metric for zero curvature and the timelike 1-form $\tilde{K}$ is given by

$$\tilde{K} = C_0\, dt + \sum_{i=1}^{N} C_i\, d\phi_i + \sqrt{h(r)}\, g^{-2} r^{-2}\, dr\,, \quad (4.37)$$

in which C_0 and C_i are some arbitrary $N + 1$ constants and the functions $(H, h(r))$ are defined to be

$$H = 1 + 2ms^2 r^{3-D}\,, \quad h(r) = C_0^2 - g^2 \left(r^2 + \sum_{i=1}^{N} C_i^2 \right). \quad (4.38)$$

To see whether the solution (4.36) describes a regular black hole, one can perform the following coordinate transformations:

$$dt \to dt + \frac{2ms^2 C_0}{g^2 r^{D-1} \Delta} \sqrt{h(r)}\, dr\,,$$
$$d\phi_i \to d\phi_i - \frac{2ms^2 C_i}{r^{D-1} \Delta} \sqrt{h(r)}\, dr\,. \quad (4.39)$$

Then the metric and the gauge potential become

$$ds^2 = H^{1/(D-2)} \left(- g^2 r^2\, dt^2 + \frac{dr^2}{\Delta} + r^2 d\Sigma_k^2 \right.$$
$$\left. + \frac{2ms^2}{r^{D-3} H} K^2 \right), \quad (4.40)$$
$$A = \frac{2ms^2}{r^{D-3} H} K\,, \quad K = C_0\, dt + \sum_{i=1}^{N} C_i\, d\phi_i$$

where

$$\Delta = g^2 r^2 - 2ms^2 r^{3-D} h(r)\,. \quad (4.41)$$

The horizon is determined by $\Delta = 0$ and is endowed with a planar topology.

V. CONCLUSIONS

In this paper, we have studied a new metric ansatz dubbed as the stringy Kerr-Schild ansatz since it can been seen as the most meaningful generalization of the Kerr-Schild form for the (un)gauged supergravity theory, in which the general black hole solutions in all dimensions share a common and universal metric structure.

Initially, we have adopted the traditional method of perturbation expansion for this new ansatz, because it had already been successfully applied in the extended KS ansatz [30]. We attempted to explore some properties of the sKS form in the usual way. But unfortunately, as a consequence, the introduction of the perturbation factor appears clearly to be in contradiction with the basic assumption of the sKS ansatz. Therefore, the traditional perturbation expansion analysis of the full metric tensor provides no help to reach our aim.

As such, we have proposed a new method of background metric expansion to extract simple information by expanding the field equations of the full spacetimes around the background metric. In Sec. III, we obtained the geodesic condition (3.14) from the Lagrangian of the Einstein-Maxwell-dilaton system with respect to the background metric and the counterparts (3.27), (3.28), and (3.29) of all the field equations around the background metric. The condition (3.27) depends only on the properties of the dilaton scalar Φ, and what is more, the set of the sufficient conditions (3.27), (3.28), and (3.29) is satisfied spontaneously in the uncharged case; thus our method coincides with the usual method of perturbation expansion for the Kerr-Schild form. As anticipated, the overall calculations can be substantially simplified in our method; meanwhile, the results of our analysis could be helpful in obtaining new exact solutions of the sKS form. As two examples of applications of our method, we first rederived the rotating single-charged KK-AdS solutions by further introducing one or two arbitrary constants (corresponding to even and odd dimensions, respectively). Moreover, for further verification, we obtained new solutions by using the planar anti-de Sitter metric as the background one in sKS ansatz.

It is worthwhile to investigate whether the analysis made in this paper can be generalized to multiple-charged black hole solutions [44, 45] in supergravity theories, since the general nonextremal rotating charged AdS black hole solutions with two independent charge parameters still remain elusive in $D = 6, 7$ gauged supergravities.

Appendix A: Double Kerr-Schild form for $D = 4$ Kerr-NUT-AdS solution

The four-dimensional Kerr-NUT-AdS metric admits a double Kerr-Schild representation as follows:

$$ds^2 = -\frac{(1 + g^2 r^2)(1 - g^2 y^2)}{\chi} dt^2 + \frac{(r^2 + y^2)\, dr^2}{(r^2 + a^2)(1 + g^2 r^2)} + \frac{(r^2 + y^2)\, dy^2}{(a^2 - y^2)(1 - g^2 y^2)} + \frac{(r^2 + a^2)(a^2 - y^2)}{a^2 \chi} d\phi^2$$
$$+ \frac{2mr}{r^2 + y^2} K^2 + \frac{2ny}{r^2 + y^2} N^2,$$

where two null 1-forms are

$$K = \frac{1 - g^2 y^2}{\chi} dt - \frac{a^2 - y^2}{a\chi} d\phi - \frac{(r^2 + y^2)\, dr}{(r^2 + a^2)(1 + g^2 r^2)}, \qquad N = \frac{1 + g^2 r^2}{\chi} dt - \frac{r^2 + a^2}{a\chi} d\phi - \frac{i(r^2 + y^2)\, dy}{(a^2 - y^2)(1 - g^2 y^2)}.$$

The vectors K_μ and N_μ are two *linearly independent*, mutually orthogonal, affinely-parameterized null geodesic congruences, so they need not be proportional to each other. Note that N_μ is a complex vector rather than a real vector.

The most general Plebański-Demiański type-D solution [12] with an extra acceleration parameter can be put into a similar form. Higher-dimensional generalizations with just one rotation parameter were presented in Ref. [16], while the multi-Kerr-Schild form [17] has been studied in detail for the most general AdS solution with NUT charges.

[1] R.P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).

[2] R.P. Kerr and A. Schild, Proc. Symp. Appl. Math. **17**,

199 (1965).

[3] R.P. Kerr and A. Schild, Gen. Rel. Grav. **41**, 2485 (2009).

[4] D. Bini, A. Geralico, and R.P. Kerr, Int. J. Geom. Meth. Mod. Phys. **7**, 693 (2010).

[5] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations*, 2nd Ed. (Cambridge University Press, Cambridge, 2003).

[6] A.H. Taub, Ann. Phys. **134**, 326 (1981).

[7] J. Martin and J.M.M. Senovilla, J. Math. Phys. **27**, 265 (1986) [J. Math. Phys. **27**, 2209 (1986)].

[8] B.C. Xanthopoulos, J. Math. Phys. **19**, 1607 (1978).

[9] B.C. Xanthopoulos, Classical Quantum Gravity **3**, 157 (1986).

[10] L.Á. Gergely, Classical Quantum Gravity **19**, 2515 (2002).

[11] J.F. Plebański, Ann. Phys. (N.Y.) **90**, 196 (1975).

[12] J.F. Plebański and M. Demiański, Ann. Phys. (N.Y.) **98**, 98 (1976).

[13] J.F. Plebański and A. Schild, Nuovo Cimento Soc. Ital. Fis **35B**, 35 (1976).

[14] G.S. Hall, M.S. Hickman, and C.B.G. McIntosh, Gen. Rel. Grav. **17**, 475 (1985).

[15] C.B.G. McIntosh and M.S. Hickman, Gen. Rel. Grav. **20**, 793 (1988).

[16] Z.W. Chong, G.W. Gibbons, H. Lü, and C.N. Pope, Phys. Lett. B **609**, 124 (2005).

[17] W. Chen and H. Lü, Phys. Lett. B **658**, 158 (2008).

[18] J.K. Kowalczyński and J.F. Plebański, Acta Phys. Pol. B **8**, 169 (1977).

[19] M. Štefaník and J. Horský, Acta Phys. Slov. **50**, 213 (2000).

[20] N.V. Mitskievichy and J. Horský, Classical Quantum Gravity **13**, 2603 (1996).

[21] R.C. Myers and M.J. Perry, Ann. Phys. **172**, 304 (1986).

[22] G.W. Gibbons, H. Lü, D.N. Page, and C.N. Pope, Phys. Rev. Lett. **93**, 171102 (2004).

[23] G.W. Gibbons, H. Lü, D.N. Page, and C.N. Pope, J. Geom. Phys. **53**, 49 (2005).

[24] W. Chen, H. Lü, and C.N. Pope, Classical Quantum Gravity **23**, 5323 (2006).

[25] M. Ortaggio, V. Pravda, and A. Pravdová, Classical Quantum Gravity **26**, 025008 (2009).

[26] M. Ortaggio, V. Pravda, and A. Pravdová, AIP Conf. Proc. **1122**, 364 (2009).

[27] T. Málek and V. Pravda, Classical Quantum Gravity **28**, 125011 (2011).

[28] T. Málek and V. Pravda, J. Phys. Conf. Ser. **314**, 012111 (2011).

[29] A.N. Aliev and D.K. Ciftci, Phys. Rev. D **79**, 044004 (2009).

[30] B. Ett and D. Kastor, Classical Quantum Gravity **27**, 185024 (2010).

[31] Z.W. Chong, M. Cvetič, H. Lü, and C.N. Pope, Phys. Rev. Lett. **95**, 161301 (2005).

[32] T. Málek, Classical Quantum Gravity **31**, 185013 (2014); J. Phys. Conf. Ser. **600**, 012040 (2015).

[33] T. Málek, Ph.D thesis, arXiv:1204.0291 [gr-qc].

[34] S. Bonanos, Classical Quantum Gravity **9**, 697 (1992).

[35] S.Q. Wu, Phys. Rev. D **83**, 121502(R) (2011).

[36] W.C. dos Santos, Astrophys. Space Sci. **331**, 295 (2011).

[37] J.D. Bekenstein, Phys. Rev. D **70**, 083509 (2004).

[38] M. Novello and E. Bittencourt, Phys. Rev. D **86**, 124024 (2012).

[39] B. Chen and R. Kantowski, Phys. Rev. D **79**, 104007 (2009).

[40] V. Baccetti, P. Martin-Moruno, and M. Visser, J. High Energy Phys. 08 (2012) 108.

[41] A. Awad, Classical Quantum Gravity **20**, 2827 (2003).

[42] K. Peeters, Comput. Phys. Commun. **176**, 550 (2007).

[43] K. Peeters, hep-th/0701238.

[44] S.Q. Wu, Phys. Lett. B **705**, 383 (2011).

[45] S.Q. Wu, Phys. Lett. B **707**, 286 (2012).